CHAPTER 5: Concave and Quasiconcave Functions

1 Concave and Convex Functions

1.1 Definitions and Properties

Let $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function. From now on we will assume that X is a convex subset of $\mathbb{R}^n$. The *hypograph* and *epigraph* of f are defined as follows.

Definition 1 (Hypograph and epigraph of a function) *The hypograph (or subgraph) and epigraph of a function f, denoted respectively $hyp\, f$ and $epi\, f$, are defined as*

$$\begin{aligned} hyp\, f &:= \{(x, y) \in X \times \mathbb{R} | f(x) \geq y\}, \\ epi\, f &:= \{(x, y) \in X \times \mathbb{R} | f(x) \leq y\}. \end{aligned}$$

Intuitively, the hypograph of a function is the area lying below the graph of the function, while the epigraph is the area lying above the graph.

Definition 2 (Concave and convex functions) *A function $f : X \longrightarrow \mathbb{R}$ is concave (convex) on X if $hyp\, f$ ($epi\, f$) is convex.*

The following theorem gives a characterization of concave (convex) functions.

Theorem 1 *A function $f : X \longrightarrow \mathbb{R}$ is concave (convex) on X if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, it is the case that*

$$f(\lambda x + (1-\lambda)y) \geq (\leq) \lambda f(x) + (1-\lambda) f(y).$$

Proof First, suppose that f is concave, i.e., $hyp\, f$ is convex. Let x and y be two arbitrary points in X. Then, $(x, f(x)) \in hyp\, f$ and $(y, f(y)) \in hyp\, f$. Since $hyp\, f$ is convex, we have, for any $\lambda \in [0, 1]$,

$$(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) \in hyp\, f.$$

By definition of $hyp\, f$, a point (w, z) is in $hyp\, f$ only if $f(w) \geq z$, and hence

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y).$$

Now suppose that for all $x, y \in X$ and for all $\lambda \in [0, 1]$, it is the case that

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

We will show that for two arbitrary points (w_1, z_1) and (w_2, z_2) in $hyp\, f$, their convex combination is also in $hyp\, f$. Since the above two points are in $hyp\, f$, we have $f(w_1) \geq z_1$ and $f(w_2) \geq z_2$. Condition (1) and $\lambda \in [0, 1]$ imply that

$$\begin{aligned} f(\lambda w_1 + (1-\lambda) w_2) &\geq \lambda f(w_1) + (1-\lambda) f(w_2) \\ &\geq \lambda z_1 + (1-\lambda) z_2. \end{aligned}$$

Therefore, the point $(\lambda w_1 + (1-\lambda) w_2, \lambda z_1 + (1-\lambda) z_2)$ is in $hyp\, f$, and hence it is convex. The proof for a convex function is similar. ■

For strict concavity and convexity replace '$\geq$' ('$\leq$') by '$>$' ('$<$'), and take $\lambda \in (0, 1)$. The notions of concavity and convexity are neither exhaustive nor mutually exclusive. We may have functions that are neither concave nor convex, and functions that are both concave and convex. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^3$ is neither concave nor convex. To see this take $x = -2$ and $y = 2$. Then check the conditions of the above theorem for $\lambda = 1/4$ and $\lambda = 3/4$. A linear function such as $f(x) = 2 + 3x$ is both concave and convex. Such functions are called *affine functions*. Another observation is that a function $f : X \longrightarrow \mathbb{R}$ is concave if and only if $-f$ is convex. The following theorem states an important result associated with concave functions.

Theorem 2 (Local-global theorem) *Let $f : X \longrightarrow \mathbb{R}$ be a concave function on X. Then*

(a) *every local maximum of f is a global maximum,*

(b) *the set* $\arg\max\{f(x) | x \in X\}$*, the set of maximizers of f on X is either empty or convex*

Proof (a) Let x^0 be any local maximum of f, but not a global maximum. Then there is $r > 0$ such that $f(x) \leq f(x^0)$ for all $x \in B_r(x^0) \cap X$. Since x^0 is not a global maximum, there is $y \in X$ such that $f(y) > f(x^0)$. Since X is convex, for any $\lambda \in (0, 1)$, $(1-\lambda)y + \lambda x^0 \in X$. Pick λ close to 1 so that $(1-\lambda)y + \lambda x^0 \in B_r(x^0)$. By concavity of f,

$$f((1-\lambda)y + \lambda x^0) \geq (1-\lambda) f(y) + \lambda f(x^0) > f(x^0)$$

since $f(y) > f(x^0)$. But $(1-\lambda)y + \lambda x^0 \in B_r(x^0)$ by construction, so $f(x^0) \geq f((1-\lambda)y + \lambda x^0) > f(x^0)$, which is a contradiction.

(b) Suppose that x_1 and x_2 both maximize f on X, i.e., $f(x_1) = f(x_2)$. By concavity of f we have, for any $\lambda \in (0, 1)$,

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2) = f(x_1).$$

The above must hold with equality or x_1 and x_2 would not be maximizers. Thus, the set of maximizers must is convex. ■

Following are some useful properties of a concave function.

Lemma 1 *Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two concave functions. Then*

(a) *the function $\alpha f + \beta g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is concave for $\alpha, \beta \geq 0$,*

(b) *the function* $\min\{f, g\} : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is concave,*

(c) *if* $h:\mathbb{R}\longrightarrow\mathbb{R}$ *is non-decreasing and concave, then* $h\circ f:\mathbb{R}^n\longrightarrow\mathbb{R}$ *is concave.*

Proof The proof of the above lemma is left as an exercise. ■

Another important implication of concavity is stated in the following lemma.

Lemma 2 *Let* $f:X\longrightarrow\mathbb{R}$ *be a concave function on* X*. For any* $\alpha\in\mathbb{R}$*, the upper contour set of* f*,*

$$U_f(\alpha):=\{x\in X|\,f(x)\geq\alpha\}$$

is either empty or convex. Similarly, if f *is convex, then the lower contour set of* f*,*

$$L_f(\alpha):=\{x\in X|\,f(x)\leq\alpha\}$$

is either empty or convex.

Proof Let x' and x'' be two points in $U_f(\alpha)$, i.e., $f(x')\geq\alpha$ and $f(x'')\geq\alpha$. By concavity of f,

$$f(\lambda x'+(1-\lambda)x'')\geq\lambda f(x')+(1-\lambda)f(x'')\geq\lambda\alpha+(1-\lambda)\alpha=\alpha.$$

Therefore, $\lambda x'+(1-\lambda)x''$ is in $U_f(\alpha)$. ■

The converse statement of the above theorem is not always true. To guarantee the sufficiency we would need a weaker condition called quasiconcavity, which will be introduced later.

1.2 Continuity of Concave Functions

The main result of this subsection is that a concave function must be continuous everywhere on its domain, except perhaps at the boundary points.

Theorem 3 *Let* $f:X\longrightarrow\mathbb{R}$ *be a concave function on* X*. Then* f *is continuous on* $int(X)$*.*

Proof See Sundaram (1996, pp. 177). ■

Continuity of a concave function may fail in the boundary of X. Consider the following example.

Example 1 Define $f:[0,1]\longrightarrow\mathbb{R}$ by

$$f(x)=\begin{cases}\sqrt{x} & \text{if } 0<x<1,\\ -1 & \text{if } x=0,\,1.\end{cases}$$

Then f is strictly concave on [0, 1], but is discontinuous at 0 and 1. □

1.3 Differentiable Concave Functions

Concave functions have nice characterization when they are differentiable.

Theorem 4 *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ is a $\mathscr{C}^1$ function. Then f is concave if and only if for all x^0, $x \in X$, it is the case that*

$$f(x) \leq f(x^0) + \nabla f(x^0) \cdot (x - x^0).$$

Proof Suppose that f is concave on X. Take x, $x^0 \in X$. By the concavity of f, we have, for all $\lambda \in [0, 1]$,

$$\begin{aligned} & f(x^0 + \lambda(x - x^0)) \geq f(x^0) + \lambda[f(x) - f(x^0)] \\ \Rightarrow \quad & \frac{f(x^0 + \lambda(x - x^0)) - f(x^0)}{\lambda} \geq f(x) - f(x^0). \end{aligned}$$

Taking the limit of the above expression as $\lambda \to 0^+$, we get

$$\nabla f(x^0) \cdot (x - x^0) \geq f(x) - f(x^0).$$

To prove the converse, take x' and x'' in X, and let $x^\lambda = \lambda x' + (1 - \lambda)x''$. By hypothesis,

$$\begin{aligned} f(x') &\leq f(x^\lambda) + \nabla f(x^\lambda) \cdot (x' - x^\lambda), \\ f(x'') &\leq f(x^\lambda) + \nabla f(x^\lambda) \cdot (x'' - x^\lambda) \end{aligned}$$

Multiplying the first inequality by λ and the second inquality by $1 - \lambda$, and adding we get

$$f(x^\lambda) \geq \lambda f(x') + (1 - \lambda) f(x'').$$

This completes the proof. ■

Theorem 5 *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ is a $\mathscr{C}^2$ function. Then f is concave if and only if $\nabla^2 f(x)$ is negative semi-definite for all $x \in X$.*

Proof First, recall that a symmetric $n \times n$ matrix A is negative semi-definite if and only if $h^T A h \leq 0$ for all $h \in \mathbb{R}^n$. Suppose that f is concave. Take an $x \in X$ and an arbitrary direction vector h in $\mathbb{R}^n$. Because X is open, there is some $\delta > 0$ such that $x + \alpha h \in X$ for all $\alpha \in I \equiv (-\delta, \delta)$. Define $g : I \longrightarrow \mathbb{R}$ by

$$g(\alpha) := f(x + \alpha h) - f(x) - \nabla f(x) \alpha h. \tag{2}$$

Since f is $\mathscr{C}^2$, g is also $\mathscr{C}^2$ with $g(0) = 0$. By the previous theorem, the concavity of f implies

$$f(x + \alpha h) \leq f(x) + \nabla f(x) \alpha h \;\Rightarrow\; g(\alpha) \leq 0, \text{ for all } \alpha \in I.$$

Thus, g is a twice continuously differentiable function with a maximum at 0, and hence we must have $g'(0) = 0$ and $g''(0) \leq 0$. Now differentiating twice (2) with respect to α we get

$$g''(\alpha) = h^T \nabla^2 f(x + \alpha h)\, h.$$

And $g''(0) \leq 0$ implies that

$$g''(0) = h^T \nabla^2 f(x)\, h \leq 0.$$

Since the vector is chosen arbitrarily, we conclude that $\nabla^2 f(x)$ is negative semi-definite for all $x \in X$.

To show the converse, suppose that $\nabla^2 f(x)$ is negative semi-definite. Pick any two points x and $x+h$ in X. By Taylor's theorem, we have, for some $\alpha \in (0, 1)$,

$$f(x+h) - f(x) - \nabla f(x)\, h = \frac{1}{2} h^T \nabla^2 f(x+\alpha h) h,$$

where $x + \alpha h \in (x, x+h)$. Since $\nabla^2 f(x)$ is negative semi-definite, the right-hand-side of the above equation is negative, which implies

$$f(x+h) \leq f(x) + \nabla f(x)\, h,$$

and hence by the previous theorem, f is concave. ■

2 Quasiconcave and Quasiconvex Functions

2.1 Definitions and Properties

Definition 3 (Quasiconcave and quasiconvex functions) *A function $f : X \longrightarrow \mathbb{R}$ is quasiconcave on X if the upper contour set of f, $U_f(\alpha) = \{x \in X | f(x) \geq \alpha\}$ is convex. The function f is quasiconvex if the lower contour set of f, $L_f(\alpha) = \{x \in X | f(x) \leq \alpha\}$ is convex.*

The following theorem gives a characterization of quasiconcave (quasiconvex) functions.

Theorem 6 *A function $f : X \longrightarrow \mathbb{R}$ is quasiconcave on X if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, it is the case that*

$$f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}.$$

Similarly, A function $f : X \longrightarrow \mathbb{R}$ is quasiconvex on X if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, it is the case that

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}.$$

Proof First suppose that $U_f(\alpha)$ is convex. Take $x, y \in X$ and $\lambda \in [0, 1]$. Assume without loss of generality that $f(x) \geq f(y) = \alpha$. Thus, $x, y \in U_f(\alpha)$, and by convexity of $U_f(\alpha)$ we have $\lambda x + (1-\lambda) y \in U_f(\alpha)$, which means

$$f(\lambda x + (1-\lambda) y) \geq \alpha = f(y) = \min\{f(x), f(y)\}.$$

To show the converse, suppose that $f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}$ for all $x, y \in X$ and for all $\lambda \in [0, 1]$. If $U_f(\alpha)$ is empty or singleton, then it is trivially convex. So suppose that $U_f(\alpha)$ contains at least two points x and y. Then $f(x) \geq \alpha$ and $f(y) \geq \alpha$, and so $\min\{f(x), f(y)\} \geq \alpha$. By hypothesis, $f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}$, and hence $\lambda x + (1-\lambda) y \in U_f(\alpha)$. Since α was chosen arbitrarily, this completes the proof of the first part. The proof of the second part is analogous. ■

For strict quasiconcavity and quasiconvexity replace '$\geq$' ('$\leq$') by '$>$' ('$<$'), and take $\lambda \in (0, 1)$. Also, a function $f : X \longrightarrow \mathbb{R}$ is (strictly) quasiconcave if and only if $-f$ is (strictly) quasiconvex. The notions of quasiconcavity and quasiconvexity are generalizations of concavity and convexity, respectively. For instance, the set of all concave functions are contained in the set of quasiconcave functions.

Theorem 7 *Let $f : X \longrightarrow \mathbb{R}$ be concave on X, then it is also quasiconcave. Similarly, if $f : X \longrightarrow \mathbb{R}$ be convex on X, then it is also quasiconvex.*

Proof The concavity of f implies that, for all $x, y \in X$ and for all $\lambda \in [0, 1]$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda) f(y) \\ &\geq \lambda \min\{f(x), f(y)\} + (1-\lambda)\min\{f(x), f(y)\} \\ &= \min\{f(x), f(y)\}. \end{aligned}$$

So f is quasiconcave. ■

Following lemma states some useful properties of quasiconcave functions.

Lemma 3 *Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two quasiconcave functions. Then*

(a) *the function $\alpha f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconcave for $\alpha \geq 0$,*

(b) *the function $\min\{f, g\} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconcave,*

(c) *if $h : \mathbb{R} \longrightarrow \mathbb{R}$ is non-decreasing, then $h \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconcave.*

Proof The proof of the above lemma is left as an exercise. ■

With respect to the above lemma, $f + g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is not necessarily quasiconcave. Suppose that $f(x) = x^3$ and $g(x) = -x$ are both quasiconcave, but $f(x) + g(x) = x^3 - x$ is not quasiconcave. Theorem 7 and Lemma 3(c) together imply that monotonic transformation of any concave function would result in a quasiconcave function. But the converse, i.e., whether any quasiconcave function is a monotonic transformation of a concave function, is not necessarily true. That is why quasiconcavity is a generalization of concavity. Consider the following example.

Example 2 Define $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1, \\ (x-1)^2, & \text{if } x > 1. \end{cases}$$

Since f is non-decreasing, it is quasiconcave. Now, suppose that there existed a concave function g and a strictly increasing function h such that $h \circ g \equiv f$. First, observe that g must be constant on $[0, 1]$. Suppose that, for $x, y \in [0, 1]$, we have $x > y \Rightarrow g(x) > g(y)$. Since h is strictly increasing, we must have $f(x) = h(g(x)) > h(g(y)) = f(y)$, which is a contradiction to the fact that f is constant on $[0, 1]$. Next, observe that g must be strictly increasing in x for $x > 1$. Apply an argument similar to the above to establish this fact. Since g is constant on $[0, 1]$, it has a local maximum at every $x^* \in (0, 1)$. These local maxima are not global maxima since g is strictly increasing for $x > 1$. This contradicts the fact that every local maximum of a concave function is a global maximum. □

Quasiconcave functions do not have similar implications for continuity and differentiability as concave function. To see this consider the following example.

Example 3 Define $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^3, & \text{if } 0 \le x \le 1, \\ 1, & \text{if } 1 < x \le 2, \\ x^3 & \text{if } x > 2. \end{cases}$$

Since f is non-decreasing, it is both quasiconcave and quasiconvex on $\mathbb{R}$. But f is discontinuous at $x = 2$. Moreover, f is constant on $(1, 2)$, and hence every point in this open interval is a local maximum as well as a local minimum. However, no point in $(1, 2)$ is either a global maximum nor a global minimum. Finally, $f'(0) = 0$, but 0 is neither a local maximum nor a local minimum. □

2.2 Differentiable Quasiconcave Functions

Quasiconcave and quasiconvex functions have derivative characterizations as do the concave and convex functions.

Theorem 8 *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ is a $\mathscr{C}^1$ function. Then f is quasiconcave if and only if for any x, $y \in X$, it is the case that*

$$f(y) \ge f(x) \Rightarrow \nabla f(x) \cdot (y - x) \ge 0.$$

Proof First, suppose that f is quasiconcave on X, and let x, $y \in X$ such that $f(y) \ge f(x)$. Let $\lambda \in (0, 1)$. Since f is quasiconcave, we have

$$\begin{aligned} & f(x + \lambda(y - x)) \ge \min\{f(x), f(y)\} = f(x) \\ \Rightarrow\ & \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \ge 0, \text{ for all } \lambda \in (0, 1). \end{aligned}$$

Taking the limit of the above expression as $\lambda \to 0^+$ we get $\nabla f(x) \cdot (y - x) \ge 0$.

To show the converse, define by $g(\lambda) := f(x + \lambda(y - x))$. Observe that $g(0) = f(x) \le f(y) = g(1)$, and that g is a $\mathscr{C}^1$ function on $[0, 1]$ with the derivative $g'(\lambda) = \nabla f(x + \lambda(y - x)) \cdot (y - x)$. Now suppose that, for all x, $y \in X$, we have $f(y) \ge f(x) \Rightarrow \nabla f(x) \cdot (y - x) \ge 0$. We will show that this implies the quasiconcavity of f, i.e.,

$$g(\lambda) = f(x + \lambda(y - x)) \ge \min\{f(x), f(y)\} = f(x) = g(0), \text{ for all } \lambda \in [0, 1].$$

Suppose on the contrary that the above is not true, i.e., there is some $\lambda^0 \in (0, 1)$ such that $g(\lambda^0) < g(0)$. Because $g(0) \le g(1)$, we can chose λ^0 such that $g'(\lambda^0) > 0$. Now let $z^0 = x + \lambda^0(y - x)$. Because $f(x) = g(0) > g(\lambda^0) = f(z^0)$ and $z^0 \in X$, we have, by hypothesis, that

$$\nabla f(z^0) \cdot (x - z^0) = \nabla f(z^0) \cdot (-\lambda^0)(y - x) \ge 0 \Rightarrow \nabla f(z^0) \cdot (y - x) \le 0. \tag{3}$$

The above is true since $\lambda^0 > 0$. On the other hand, we have chosen λ^0 such that

$$g'(\lambda^0) = \nabla f(z^0) \cdot (y - x) > 0. \tag{4}$$

Inequalities (3) and (4) contradict each other. Therefore, there cannot be such $\lambda^0 \in (0, 1)$ with $g(\lambda^0) < g(0)$, and we conclude that f is quasiconcave. ■

It is also possible to have a second-derivative test for quasiconcave functions as their concave counterparts. Let a $\mathscr{C}^2$ function f be defined on some domain $X \subset \mathbb{R}^n$, and let $x \in X$. For $k = 1, \ldots, n$, let $\bar{H}_k[f(x)]$ be the $(k+1) \times (k+1)$ matrix given by

$$\bar{H}_k[f(x)] := \begin{bmatrix} 0 & f_1(x) & \ldots & f_k(x) \\ f_1(x) & f_{11}(x) & \ldots & f_{1k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x) & f_{k1}(x) & \ldots & f_{kk}(x) \end{bmatrix}$$

The above matrix is the leading principal minor of the bordered Hessian of f at x of order k. The following theorem gives the second-derivative characterization of a quasiconcave function.

Theorem 9 *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \longrightarrow \mathbb{R}$ is a $\mathscr{C}^2$ function. Then*

(a) *if f is quasiconcave on X, then we have $(-1)^k\, det\left(\bar{H}_k[f(x)]\right) \geq 0$ for $k = 1, \ldots, n$;*

(b) *if $(-1)^k\, det\left(\bar{H}_k[f(x)]\right) > 0$ for $k = 1, \ldots, n$, then f is quasiconcave on X.*

Proof See Sundaram (1996, pp.217). ■

In the above theorem, the week inequality is not a sufficient condition for quasiconcavity. To illustrate the point, let $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is such that $f(x, y) = (x-1)^2(y-1)^2$. It is easy to check that

$$det(\bar{H}_1[f(x, y)]) = -4(x-1)^2(y-1)^4 \leq 0, \text{ for all } x, y \in \mathbb{R}^2_+,$$
$$det(\bar{H}_2[f(x, y)]) = 16(x-1)^4(y-1)^4 \geq 0, \text{ for all } x, y \in \mathbb{R}^2_+,$$

with equalities holding in either case if and only if $x = 1$ or $y = 1$. Notice that $f(0, 0) = f(2, 2) = 1$. Now take $\lambda = 1/2$. Then

$$f((1/2)(0, 0) + (1/2)(2, 2)) = f(1, 1) = 0 < 1 = \min\{f(0, 0), f(2, 2)\},$$

and hence f is not quasiconcave.